

# Surface Gap Solitons at a Nonlinearity Interface

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# Outline

- 1 Introduction
- 2 Background
  - Floquet Theory
  - Gap Solitons
- 3 Surface Gap Solitons - Smooth Potential (A)
  - Bifurcations of SGS's from Nondegenerate GS's
  - Numerical Computations
  - Asymptotics near GS Bifurcation Points
- 4 Surface Gap Solitons - per. potential with singularity at  $x = 0$  (B)
  - Gluing Technique
  - Numerical Homotopy Continuation
- 5 Conclusions, Future Work

# Introduction

Surface wave = wave inherent to the interface of 2 (or more) materials

Examples of nonlinear electromagnetic surface waves

- surface plasmon polaritons at the metal/dielectric interface [Kawata, 2001]
- interface of a linear and Kerr nonlinear dielectric [Maradudin, 1983; Tomlinson, 1980]
- interface of two homogenous Kerr nonlinear dielectrics  
[Maradudin, 1983; Stegeman et al., 1985; Boardman et al., 1991]
- interface of a periodic lattice and a homogenous dielectric  
[Makris et al., 2005],[EXPERIMENT:Suntsov et al., 2006]
- interface of two periodic dielectrics with saturable nonlinearity and with different  $\bar{n}_0$  [Kartashov et al., 2006]

Our setting:

- interface of two periodic media with different coefficients of Kerr nonlinearity
  - periodicity  $\Rightarrow$  gaps in the linear spectrum
  - nonlinear (de)focusing  $\Rightarrow$  solitary waves in the gaps possible (gap solitons)
  - material interface  $\Rightarrow$  special solitary waves localized at the interface

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# Introduction - Mathematical Model

## Gross-Pitaevsky equation

$$iu_t = -u_{xx} + V(x)u - \Gamma(x)|u|^2u, \quad x \in \mathbb{R}, t \geq 0$$

$$\Gamma(x) = \Gamma_+ \text{ for } x \geq 0, \Gamma_- \text{ for } x < 0$$

$V(x)$ ... real, continuous and  $\left\{ \begin{array}{l} \text{case A : } V(x) \text{ smooth, } d\text{-periodic} \\ \text{case B : } V(x) \text{ } d\text{-periodic on } x \geq 0 \text{ and on } x < 0 \\ \quad V'(x) \text{ discontin. at } x = 0, \end{array} \right.$

Choose:  $V_0(x) = \sin^2\left(\frac{\pi x}{10}\right)$

case A:  
 $V(x) = V_0(x)$

case B:  
 $V(x) = V_0(x - \delta)\chi_{(-\infty, 0)} + V_0(x + \delta)\chi_{[0, \infty)}$

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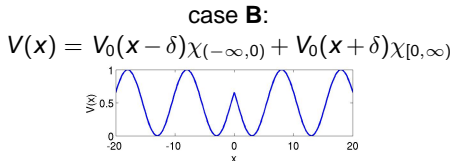
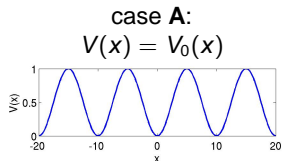
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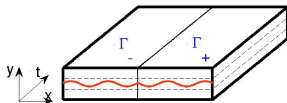
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## Applications:

- beam propagation in Kerr nonlinear dielectrics with transverse periodicity
- Bose-Einstein Condensates loaded on an optical lattice



Note: In most dielectrics a jump in the nonlinear part of  $n$   $\Rightarrow$  much larger jump in the linear part

BUT not in all materials [Blömer et al., 2006]

**Aim:** single-hump, exponentially decaying, solitary wave solutions

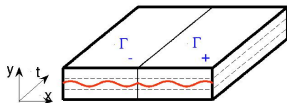
$u = e^{-i\omega t} \phi(x)$ ,  $\phi : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\phi \in C^1(\mathbb{R})$  centered at  $x = 0$

$$-\phi'' - \omega\phi + V(x)\phi - \Gamma(x)\phi^3 = 0, x \in \mathbb{R} \quad (1)$$

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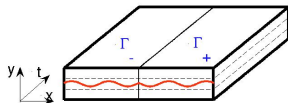
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# Linear problem for a smooth $V(x)$ (Floquet theory)

$$L\psi = \omega\psi, \quad L = -\partial_x^2 + V(x), \quad V(x+d) = V(x)$$

Hill's equation:

$$\sigma(L) = \sigma_c(L) = \Sigma = [\omega_0, \omega_1] \cup [\omega_2, \omega_3] \cup \dots$$

$$\omega_{2n-2} < \omega_{2n-1} \leq \omega_{2n}, \quad \omega_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

- $\omega \in \text{int}(\Sigma) \Rightarrow \psi$  quasi-periodic:  
 $\psi = p(x)e^{\pm ikx}, k \in [0, \pi/d], p(x+d) = p(x)$
- $\omega = \omega_n \Rightarrow$   
 one solution  $\psi_n(x)$  periodic or anti-periodic ( $k = 0$  or  $k = \pi/d$ )  
 second solution grow linearly
- $\omega \in \mathbb{R} \setminus \Sigma \Rightarrow \psi$  grows exponentially as  $x \rightarrow \infty$  or  $x \rightarrow -\infty$

$n$	0	1	2	3	4	5	6	7
symmetry	even	even	odd	odd	even	even	odd	odd
periodicity	$S_+$	$S_-$	$S_-$	$S_+$	$S_+$	$S_-$	$S_-$	$S_+$
$\text{sign}(\omega''(k))$	1	-1	1	-1	1	-1	1	-1

$S_+ \dots d$ -periodic,

$S_- \dots d$ -antiperiodic

[Eastham, 1973]

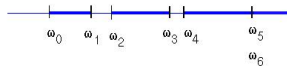
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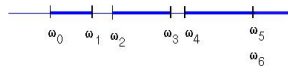
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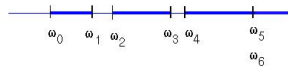
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# Gap Solitons for $\Gamma(x) \equiv \Gamma_0 = \text{const.}$

$$-\phi'' - \omega\phi + V(x)\phi - \Gamma_0\phi^3 = 0, \quad x \in \mathbb{R}$$

Gap solitons exist for all  $\omega \in \mathbb{R} \setminus \Sigma$ . [Pelinovsky et al., 2004; Pankov, 2005]

- $\omega'' > 0$  at even band edges  $\Rightarrow$  GS's bifurcate from  $\omega_{2m}$  for  $\Gamma_0 > 0$
- $\omega'' < 0$  at odd band edges  $\Rightarrow$  GS's bifurcate from  $\omega_{2m+1}$  for  $\Gamma_0 < 0$

GS example ( $\Gamma_0 = -1, \omega \in (\omega_1, \omega_2)$ ):

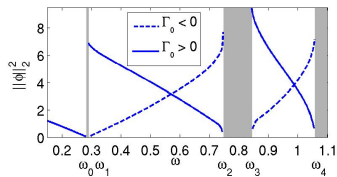
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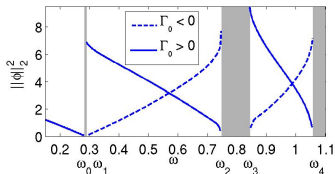
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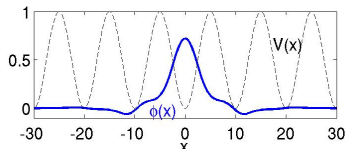
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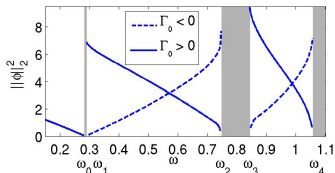
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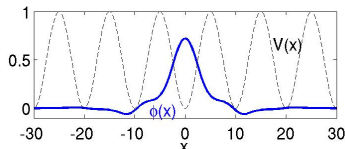
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# SGS's for a Smooth Potential $V(x)$ and $\Gamma_+ \neq \Gamma_-$

## Bifurcations of SGS's from GS's:

Rewrite  $-\phi'' - \omega\phi + V(x)\phi - \Gamma(x)\phi^3 = 0$  as

$$F(\phi, \nu, \gamma) = -\phi'' - \omega\phi + V(x)\phi - \gamma\phi^3 - \nu\text{sign}(x)\phi^3 = 0,$$

where  $\gamma = (\Gamma_+ + \Gamma_-)/2, \nu = (\Gamma_+ - \Gamma_-)/2$ .

- GS  $\phi_{(0, \gamma_0)}(x)$  given at  $\nu = 0, \gamma = \gamma_0, \omega \in \mathbb{R} \setminus \Sigma$
- $J = D_\phi F(\phi_{(0, \gamma_0)}, 0, \gamma_0) = -\partial_x^2 - \omega + V(x) - 3\gamma_0\phi_{(0, \gamma_0)}^2$
- $J \dots$  relatively compact perturb. of  $L - \omega$

$$\Rightarrow \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(L - \omega) = \Sigma - \omega$$

- $\omega \in \mathbb{R} \setminus \Sigma \Rightarrow d(0, \sigma_{\text{ess}}(J)) > 0$ .
- *nondegeneracy condition*: no zero e-vals of  $J$   
(expected as shift invariance is broken)

$\therefore$  Impl. Function Thm:  $\exists! \phi_{(\nu, \gamma)}$  for  $(\nu, \gamma)$  suff. close to  $(0, \gamma_0)$

## Conclusion:

If  $J$  is nonsingular, a nondegenerate GS can be continued to a SGS family for sufficiently small  $|\Gamma_+ - \Gamma_-|$

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$$F(\phi, \nu, \gamma) = -\phi'' - \omega\phi + V(x)\phi - \gamma\phi^3 - \nu\text{sign}(x)\phi^3 = 0,$$

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$$\Rightarrow \sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(L - \omega) = \Sigma - \omega$$

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$\therefore$  Impl. Function Thm:  $\exists! \phi_{(\nu, \gamma)}$  for  $(\nu, \gamma)$  suff. close to  $(0, \gamma_0)$

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If  $J$  is nonsingular, a nondegenerate GS can be continued to a SGS family for sufficiently small  $|\Gamma_+ - \Gamma_-|$

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- homotopy continuation from a GS via Newton iteration on a centered FD discretization
  - $\Gamma_+$  fixed (=1 or -1) and  $\Gamma_-$  deviated from  $\Gamma_+$  up to a threshold  $\Gamma_- = \Gamma_*$ , where the convergence fails

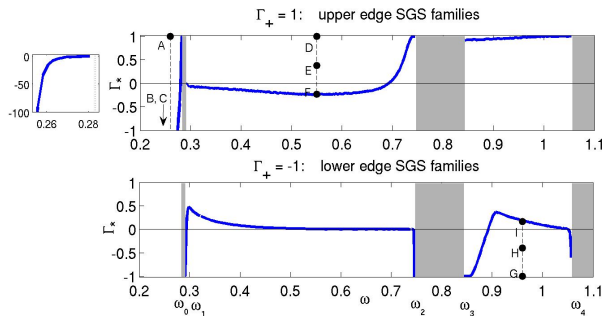
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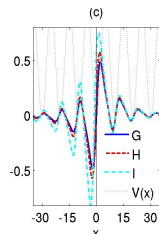
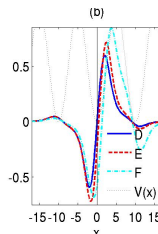
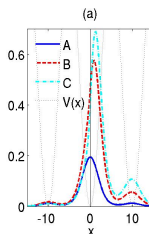
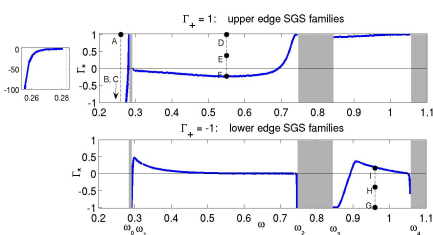
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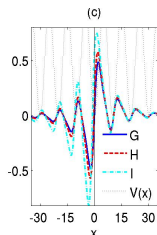
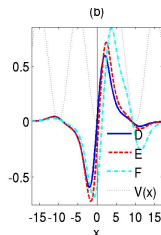
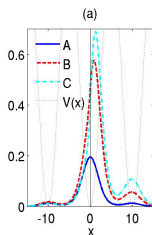
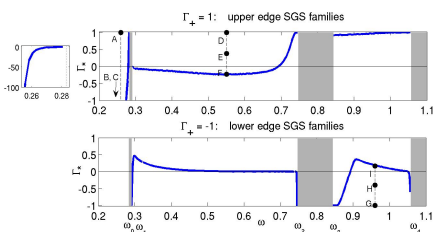
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For  $\omega$  near  $\omega_n$  GP is approximated by NLS:

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... multiple scales ...

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# Outline

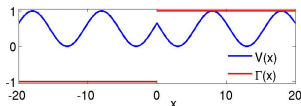
- 1 Introduction
- 2 Background
  - Floquet Theory
  - Gap Solitons
- 3 Surface Gap Solitons - Smooth Potential (A)
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  - Numerical Computations
  - Asymptotics near GS Bifurcation Points
- 4 **Surface Gap Solitons - per. potential with singularity at  $x = 0$  (B)**
  - **Gluing Technique**
  - **Numerical Homotopy Continuation**
- 5 Conclusions, Future Work

# SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

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Q: Do SGS's have anything in common with GS's on each half line?  
 A: Gluing algorithm to generate SGS's:

## -step 1- continuous solutions

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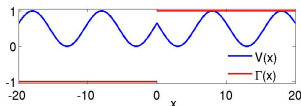
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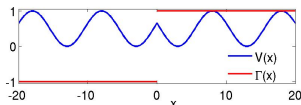
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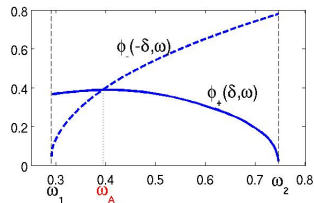
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- $\phi_{\pm}$  are continuous in  $\omega$
- $\phi_-(-\delta, \omega_{2m-1}) = \phi_+(\delta, \omega_{2m}) = 0$
- $\phi_-(-\delta, \omega_{2m}), \phi_+(\delta, \omega_{2m-1}) \neq 0$



Obtain  $\delta$ -parametrized family of continuous solutions  $\phi_{A,B}$

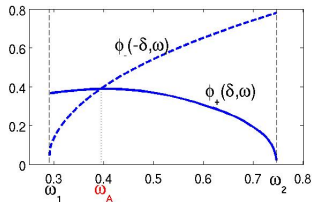
$$\phi_A(x; \delta) = \phi_-(x - \delta; \omega_A)\chi_{(-\infty, 0)} + \phi_+(x + \delta; \omega_A)\chi_{[0, \infty)}$$

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# SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

$\forall \delta > 0$   $\phi_-(-\delta, \omega)$  must intersect  $\phi_+(\delta, \omega)$  or  $-\phi_+(\delta, \omega)$  in  $(\omega_{2m-1}, \omega_{2m})$  since

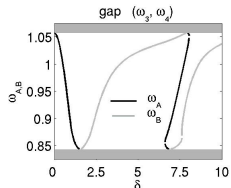
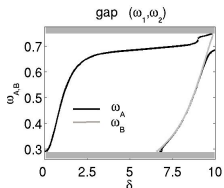
- $\phi_{\pm}$  are continuous in  $\omega$
- $\phi_-(-\delta, \omega_{2m-1}) = \phi_+(\delta, \omega_{2m}) = 0$
- $\phi_-(-\delta, \omega_{2m}), \phi_+(\delta, \omega_{2m-1}) \neq 0$



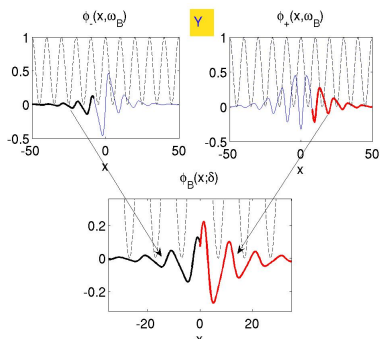
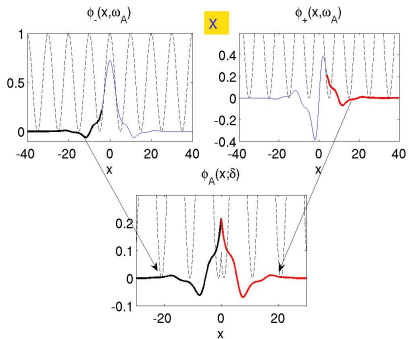
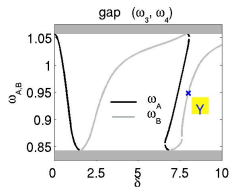
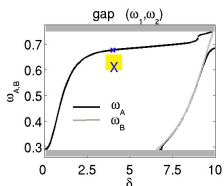
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SGS's for a Nonsmooth  $V(x)$  and  $\Gamma_+ = -\Gamma_- = 1$ 

# SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$



# SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

## -step 2- $C^1$ solutions

Search the family  $\phi_{A,B}(x; \delta)$  for  $C^1(\mathbb{R})$  solutions

Define 
$$\begin{aligned} g_A(\delta) &:= \phi'_-(-\delta; \omega_A) - \phi'_+(\delta; \omega_A) \\ g_B(\delta) &:= \phi'_-(-\delta; \omega_B) + \phi'_+(\delta; \omega_B) \end{aligned}$$
  $\delta_*$  ... zeros of either  $g_A$  or  $g_B$

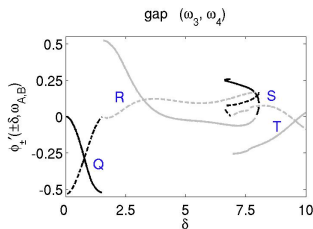
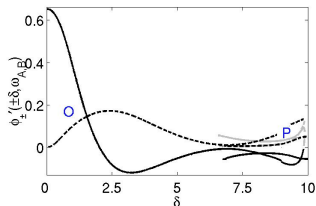
# SGS's for a Nonsmooth $V(x)$ and $\Gamma_+ = -\Gamma_- = 1$

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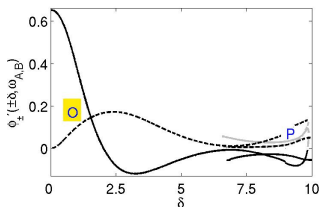
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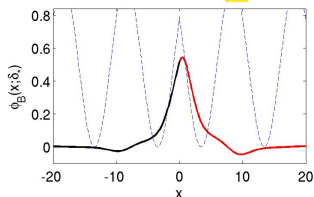
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gap  $(\omega_1, \omega_2)$

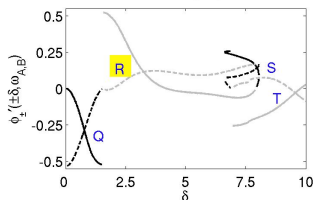


SGS at point **O**

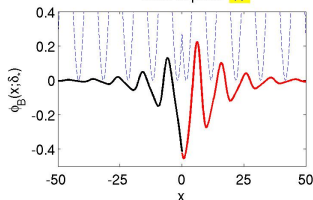


$\delta_*$  ... zeros of either  $g_A$  or  $g_B$

gap  $(\omega_3, \omega_4)$



SGS at point **R**



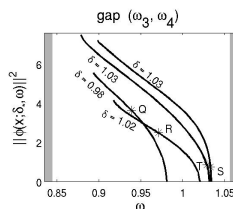
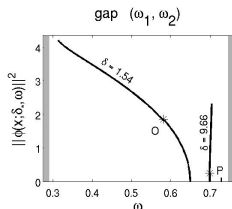
# Homotopy Continuation of $C^1$ SGSs in $\omega$

- SGS's at  $O - T$  used as starting points of a homotopy continuation in  $\omega$ :
- termination points  $O_L - T_L$  are bound states of the linear e-val problem:  

$$-\psi'' + V(x; \delta)\psi = \omega\psi, \quad V(x; \delta) = V_0(x - \delta)\chi_{(-\infty, 0)} + V_0(x + \delta)\chi_{[0, \infty)}$$

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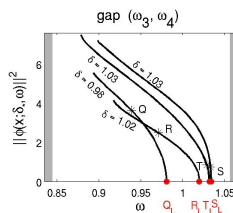
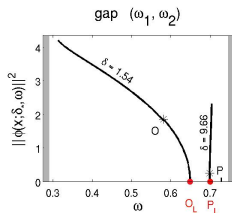


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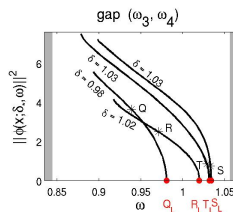
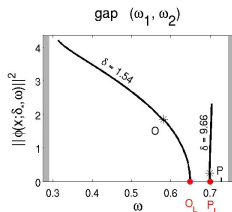


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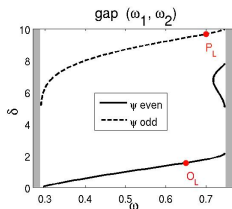
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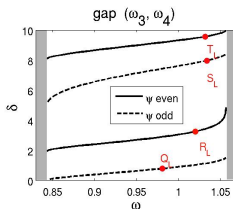


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$$\sigma_p(-\partial_x^2 + V(x; \delta))$$

for  $\delta \in (0, d)$



# Outline

- 1 Introduction
- 2 Background
  - Floquet Theory
  - Gap Solitons
- 3 Surface Gap Solitons - Smooth Potential (A)
  - Bifurcations of SGS's from Nondegenerate GS's
  - Numerical Computations
  - Asymptotics near GS Bifurcation Points
- 4 Surface Gap Solitons - per. potential with singularity at  $x = 0$  (B)
  - Gluing Technique
  - Numerical Homotopy Continuation
- 5 **Conclusions, Future Work**

# Conclusions, Future Work

## Conclusions:

- 1D SGS at a nonlinearity interface in GP eq. - smooth per. potential  $V(x)$ 
  - their bifurcation from nondegenerate GS's showed asymptotically
  - threshold nonlinearity jump computed
  - (threshold jump)  $\rightarrow 0$  as  $\omega \rightarrow$  (GS bifurcation edge)
- 1D SGS at a nonlinearity interface in GP eq. - jump in  $V'(x)$  at  $x = 0$ 
  - countably many SGS's found via gluing of 2 GS segments
  - homotopically continued into families within the frequency gaps
  - family termination points related to linear point spectrum

## Future Work:

- nonlinearity interface with discontinuous  $V(x)$  and  $V'(x)$  at  $x = 0$
- 2D, 3D:  $-\nabla^2\phi - \omega\phi + V(\vec{x})\phi - \Gamma(\vec{x})\phi^3 = 0$ ,  $\Gamma = \Gamma_{\pm}$  for  $x_1 \geq 0, x_1 < 0$ 
  - both smooth and non-smooth  $V(\vec{x})$  possible
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# Bibliography I



M.S. Eastham  
*The Spectral Theory of Periodic Differential Equations.*  
 Scottish Academic Press, 1973.



S. Kawata (ed.)  
*Near-Field Optics and Surface Plasmon Polaritons.*  
 Springer, New York, 2001.



W. J. Tomlinson  
*Surface wave at a nonlinear interface*  
*Opt. Lett.*, 5:323–325, 1980.



A.A. Maradudin  
*Nonlinear surface electromagnetic waves*  
*Optical and Acoustic Waves in Solids - Modern Topics*, M. Borissov, ed. 72–142, 1983.



G.I. Stegeman, C.T. Seaton, J. Ariyasu, R.F. Wallis and A.A. Maradudin  
*Nonlinear electromagnetic waves guided by a single interface*  
*J. App. Phys.*, 58: 2453–2459, 1985.



A.D. Boardman, P. Egan, F. Lederer, U. Langbein and M. Mihalache  
*Third-order nonlinear electromagnetic TE and TM guided waves*  
*Nonlinear Surface Electromagnetic Phenomena*, Vol. 29 of *Modern Problems in Condensed Matter Science*, H.E. Ponath and G.I. Stegeman, eds., 1991.



D.E. Pelinovsky, A.A. Sukhorukov and Y. Kivshar  
*Bifurcations and stability of gap solitons in periodic structures*  
*Physical Review E*, 70(036618):1–14, 2004.



A. Pankov  
*Periodic nonlinear Schrödinger equation with application to photonic crystals*  
*Milan J. Math.*, 73: 259–287, 2005.



Y.V. Kartashov, V.A. Vysloukh and L. Torner  
*Surface gap solitons*  
*Phys. Rev. Lett.*, 96: 073901, 2006.



K.G. Makris, S. Sunstov, D.N. Christodoulides, G.I. Stegeman, and A. Hache  
*Discrete surface solitons*  
*Opt. Lett.*, 30: 2466–2468, 2005.



D. Blömer, A. Szameit, F. Dreisow, T. Schreiber, S. Nolte, and A. Tünnermann  
*Nonlinear refractive index of fs-laser-written waveguides in fused silica*  
*Opt. Express*, 14: 2151-2157, 2006.



S. Sunstov, K. G. Makris, D. N. Christodoulides, G. I. Stegeman, A. Hache, R. Morandotti, H. Yang, G. Salamo, and M. Sorel  
*Observation of Discrete Surface Solitons*  
*Phys. Rev. Lett.*, 96: 063901, 2006.